

Overcoming the Child-Langmuir law via the magnetic mirror effect

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The maximum current in a vacuum tube prescribed by the classical Child-Langmuir law can be overcome, when the space-charge effect of the induced potential is mitigated by the mirror effect in a spatially varying magnetic field. The current could exceed the Child-Langmuir value by as much as a few factors. The regime of practical interest is examined.

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The Child-Langmuir law gives the one-dimensional maximum space-charge-limited current in a vacuum tube [1–6]. The steady state in a vacuum tube requires that the static potential energy at a given point between the electrodes does not exceed that of the cathode. In the following, we refer to this requirement as the *non-negativeness*, which is necessary because the electrons emitted from the cathode, usually of zero kinetic energy, need to reach the anode. However, at an intense enough current, the potential no longer maintains this feature as a steep curvature in the static potential gets induced by the space-charge effect, which is the origin of the Child-Langmuir law.

In this paper, the effect of a spatially varying magnetic field inside a vacuum tube is investigated, in the context of the Child-Langmuir law. Let us assume there exists an magnetic field, of which strength decreases from the cathode to the anode. If the electron emitted from the cathode initially has some perpendicular kinetic energy, the electron would gain the parallel kinetic energy as it moves toward the anode due to the magnetic moment conservation, or the magnetic mirror effect [7, 8]. The goal of this paper is to demonstrate that this effect eventually leads to overcome the Child-Langmuir law. In other words, the electrons would reach the anode even when the self-induced static potential does not satisfy the non-negativeness. In usual circumstances, the electrons emitted from the cathode do not have the perpendicular kinetic energy, however, for example, the energy could be injected into the electrons at the cathode by a microwave E&M wave via the cyclotron resonance. We estimate the physical parameters in which this effect could be of practical interest.

One-dimensional fluid equations for electrons in a vacuum tube, together with the Poisson equation, are

$$\begin{aligned}\frac{\partial n_e}{\partial t} + \frac{\partial(n_e v_x)}{\partial x} &= 0, \\ \frac{\partial v_x}{\partial t} + v_z \frac{\partial v_z}{\partial x} &= \frac{e}{m_e} \frac{\partial \phi}{\partial x},\end{aligned}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e n_e,$$

where $n_e = n_e(x)$ is the electron density between the cathode ($x = 0$) and the anode ($x = d$), $v_x = v_x(x)$ is the electron velocity, and $\phi = \phi(x)$ is the self-induced static potential. The boundary conditions are given as $v_x(0) = 0$, $\phi(0) = 0$ and $\phi(d) = V_0$, where the cathode and the anode has the bias of V_0 . At the steady state, the first two equations are simplified to $n_e(x)v_x(x) = C_1$ and $1/2m_e v_x^2(x) - e\phi(x) = C_2$, where C_1 and C_2 are constants. Then the solution can be simplified to

$$\frac{\partial^2 \phi(x)}{\partial x^2} = \frac{4\pi J}{\sqrt{\frac{2e}{m_e} \phi(x)}}, \quad (1)$$

where $J = env_x(x)$ is the current density. The maximum current maintaining the non-negativeness is

$$J_{\max} = \frac{4}{9} \left(\frac{2e}{m_e} \right)^{1/2} \frac{1}{4\pi} \frac{V_0^{3/2}}{d^2}, \quad (2)$$

which is the Child-Langmuir law.

Let us denote the magnetic field in the x -direction as $B(x)$. Consider an electron with the initial perpendicular energy $E = m_e v_\perp^2/2$ at the cathode ($x = 0$). The magnetic moment of the electron is conserved if $\omega_{ce}\delta t > 1$, where δt is the time scale with which the electrons experience a constant magnetic field and $\omega_{ce} = eB(0)/m_e c$ is the gyro-frequency. Then, the conservation of the total kinetic energy of the electron is given as

$$\frac{m_e v_x^2}{2} - e\phi(x) + \frac{m_e v_\perp^2}{2} \frac{B(x)}{B(0)} = \text{Const}, \quad (3)$$

where the last term is derived from the conservation of the magnetic moment. Eq. (3) suggests that the electron would experience the additional potential from the magnetic mirror effect:

$$e\phi_m = -\frac{m_e v_\perp^2}{2} \left(\frac{B(x)}{B(0)} - 1 \right). \quad (4)$$

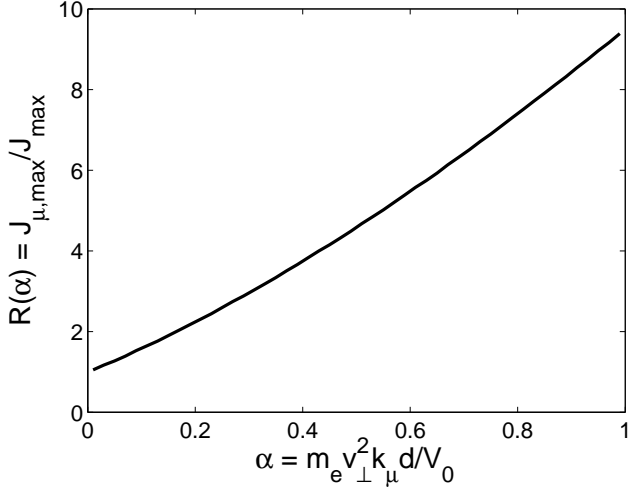


FIG. 1: The maximum current density in the presence of the additional potential from the mirror effect, scaled by that in the absence of the additional potential ($R(\alpha) = J_{\mu,\max}/J_{\max}$; see the discussion below Eq. (7)) for a range of $\alpha = m_e v_{\perp}^2 k_{\mu} d / V_0$. We assume the magnetic field profile to be $B(x) = B(0)(1 - k_{\mu}x)$ for $k_{\mu}x \leq 1$, and $B(x) = 0$ for $k_{\mu}x > 1$.

Near the cathode, Eq. (4) can be recasted as

$$e\phi_{\mu} = \frac{m_e v_{\perp}^2}{2} k_{\mu} x, \quad (5)$$

where $k_{\mu} = (dB(0)/dx)/B(0)$.

The existence of an additional time-independent potential modifies the momentum and Poisson equations to

$$\frac{1}{2} m_e v_x^2(x) - e[\phi(x) + \phi_{\mu}(x)] = \text{Const},$$

$$\frac{\partial^2 \phi(x)}{\partial x^2} = \frac{4\pi J}{\sqrt{\frac{2e}{m_e} [\phi(x) + \phi_{\mu}(x)]}}. \quad (6)$$

The non-negativeness requirement becomes

$$\phi(x) + \phi_{\mu}(x) > 0, \quad \text{for } 0 < x < d, \quad (7)$$

since $\phi(x) + \phi_{\mu}(x) > \phi(0) + \phi_{\mu}(0) = 0$. Let us denote the maximum current achievable in the absence of the additional mirror potential by J_{\max} (see Eq. (2)), and the maximum current density in the presence of the mirror potential ϕ_{μ} by $J_{\mu,\max}$. In computing $J_{\mu,\max}$, Eq. (6) is integrated for each value of $\alpha = m_e v_{\perp}^2 k_{\mu} d / V_0$ from $x = 0$ to $x = d$, using the predictor-corrector method with the boundary condition $\phi(0) = 0$ and $\phi(d) = V_0$. The maximum current $J_{\mu,\max}(\alpha)$ is determined among the current densities that do not violate Eq. (7). The ratio $R(\alpha) = J_{\mu,\max}/J_{\max}$ increases with α (Fig. 1),

In the following, the regime of practical interest is es-

timated. As shown in Fig. 1, the critical parameter is

$$\alpha = \frac{m_e v_{\perp}^2}{V_0} k_{\mu} d. \quad (8)$$

Let us consider the case where a microwave E&M wave of the same frequency as the cyclotron frequency ($\omega = eB(0)/m_e c$) injects the perpendicular kinetic energy to the electrons at the cathode. Assuming the electric field of the E&M wave is perpendicular to the magnetic field, the electron kinetic energy injected from the E&M wave can be roughly estimated to be

$$v_{\perp}^2 \cong \left(\frac{eE}{m_e \omega_{ce}} \right)^2 (\omega_{ce} \delta t_r)^2, \quad (9)$$

where δt_r is the resonance interaction time of the electron with the E&M wave and E is the electric field of the E&M wave. δt_r can be controlled by changing the spot-size of the microwave source (for instance, if the E&M wave is injected from the y -direction). However, δt_r is limited due to the fact that the magnetic field varies spatially while the frequency of the E&M wave is fixed. We estimate α from Eqs. (8) and (9) as $\alpha \cong (e^2 E^2 / m_e \omega_{ce}^2 V_0) (\omega_{ce} \delta t_r)^2$. If $\alpha > 1$, it is of plausible practical interest as shown in Fig. 1.

As an example, we consider an magnetic field of $B(0) = 1$ T. α can be estimated to be

$$\alpha \cong 0.2 \times 10^{-7} \frac{I}{V_0} (\omega_{ce} \delta t_r)^2 (k_{\mu} d), \quad (10)$$

where V_0 is in the unit of kV, and I is the intensity of the microwave in the unit of J/cm² sec. If the cathode has the area of s^2 and the spot size of the microwave is the same with the cathode area, the condition $\alpha > 1$ can be recasted as

$$P > 0.5 \times 10^7 \frac{s^2 V_0}{(\omega_{ce} \delta t_r)^2 k_{\mu} d}, \quad (11)$$

where s is in the unit of cm and $P = s^2 I$ is the power of the microwave source. If $\omega_{ce} \delta t_r \cong 10$, $d = 10$ cm, $k_{\mu} = 1/\text{cm}$, $V_0 = 1$ keV and $s^2 = 1$ cm², Eq. (11) is given as $P > 5$ kJ/sec. If $\omega_{ce} \delta t_r \cong 100$, $d = 10$ cm, $k_{\mu} = 0.1/\text{cm}$, $V_0 = 1$ keV and $s^2 = 1$ cm² Eq. (11) is given as $P > 500$ J/sec.

To summarize, a scheme overcoming the Child-Langmuir law, via the mirror effect of the magnetic moment conservation in the presence of a spatially varying magnetic field, is discussed. Due to the magnetic moment conservation, the spatially varying magnetic field generates an additional potential ϕ_{μ} on the electrons and this potential can be designed in such a way that the self-induced potential ϕ is compromised in limiting the maximum current imposed from the Child-Langmuir law. Obtaining the optimal profile of the magnetic field $B(x)$ as a function of x , which maximizes the achievable current for a given v_{\perp}^2 at the cathode, is one interesting question.

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